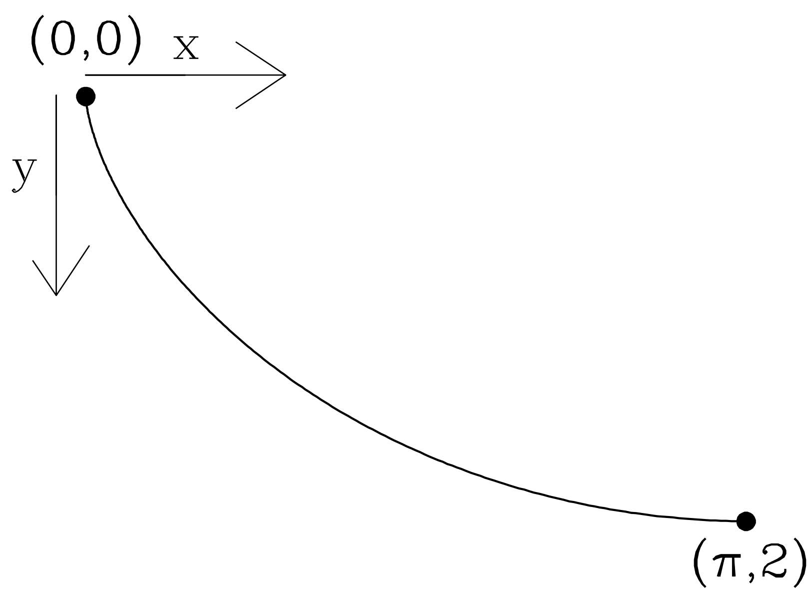
Brachistochrone

Firstname _______________ Lastname __________________

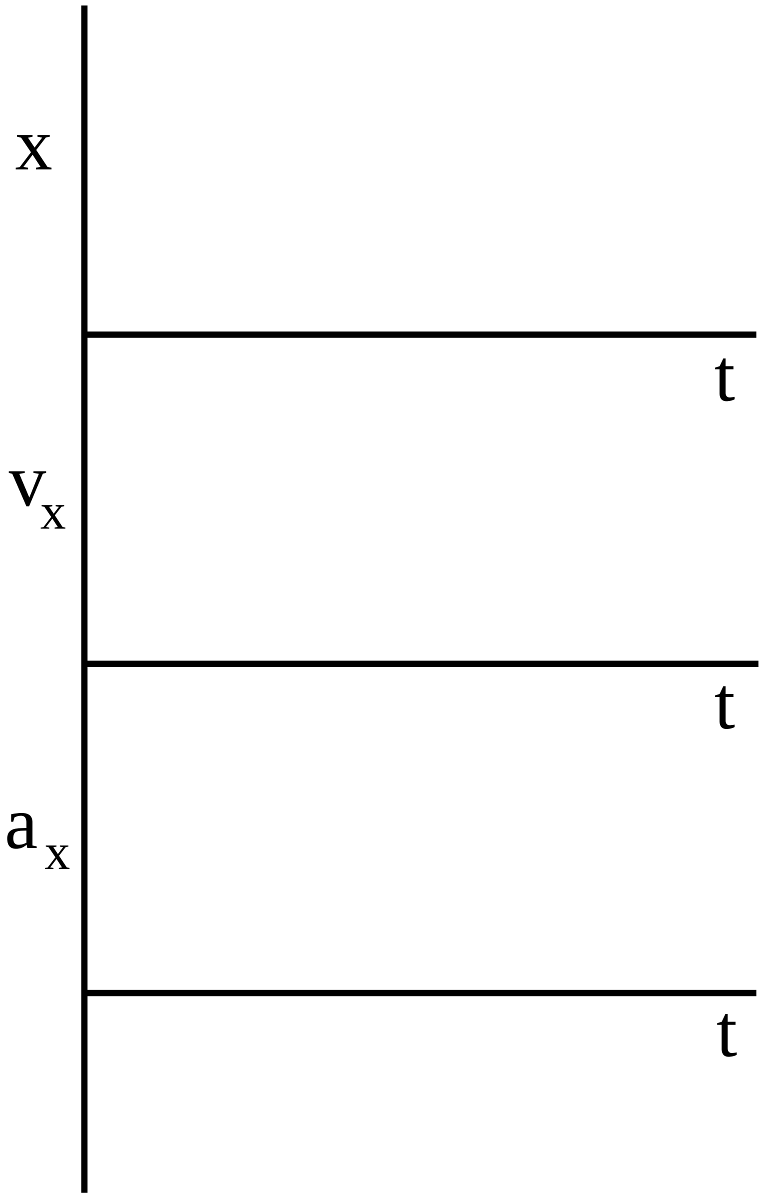
We are going to investigate a famous problem referred to as the brachistochrone. We want to have a point mass slide (without friction) between the two locations indicated by the two solid dots in the figure in the least amount of time. While we will not solve this problem here, we will show you the answer and have you think about it. The correct path is shown in the figure and it is a cycloid. This is the same shape traced by a point on the outer edge of a rolling wheel (but upside down instead). Note that the cycloid is perfectly vertical at the top and perfectly horizontal at the bottom. Also note that the coordinates are given at the beginning and the end and this makes it clear that it is flatter than a circle at the bottom. You should keep these facts in mind when answering the questions below. If there is any confusion about the very end of the path, assume that it slides up a similarly shaped brachistochrone and ends up at coordinates $(2\pi, 0)$.

While the objective historically was to find the path of least time, this solution also has an unexpected feature. The time it takes to descend to the bottom does not depend upon where you start on the cycloid. Thus, if we were to release the mass from rest anywhere along the path it would always take exactly the same time to reach the bottom.

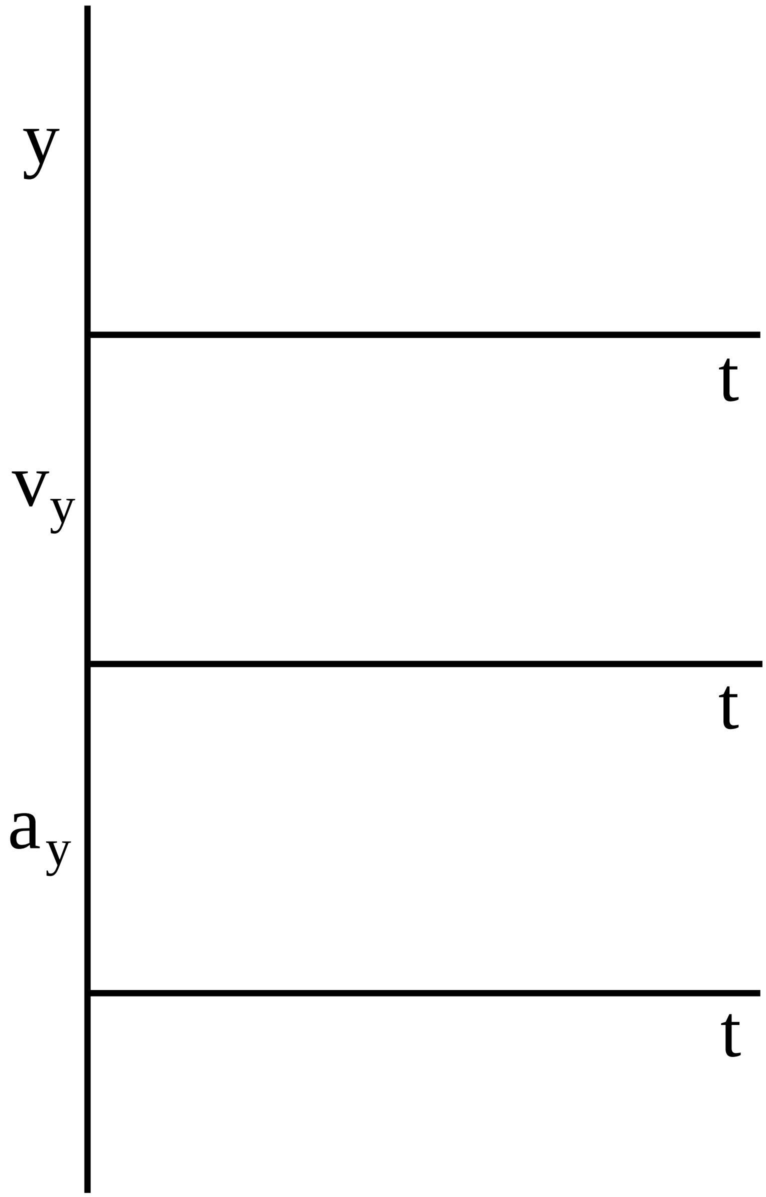


1: Explain how it is possible for the mass to start anywhere on the path above and still reach the very bottom in the exact same amount of time.

2: Graph $x(t)$, $v_x(t)$ and $a_x(t)$ qualititatively on the axes provided here and explain why each graph has the shape you have drawn. Be specific. Use the coordinate axes in the previous figure so x increases to the right and y increases down.

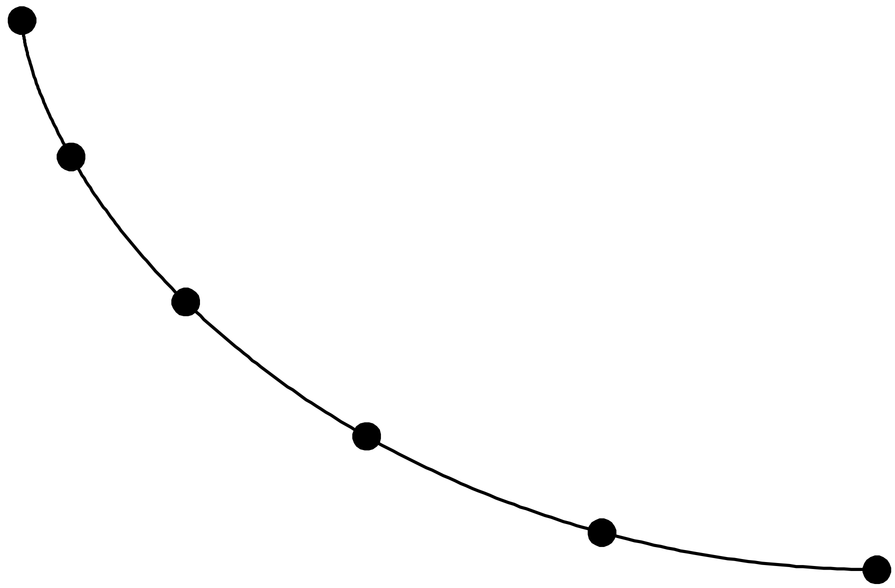


3: Graph $y(t)$, $v_y(t)$ and $a_y(t)$ qualititatively on the axes provided here and explain why each graph has the shape you have drawn. Be specific. Use the coordinate axes in the earlier figure so x increases to the right and y increases down.

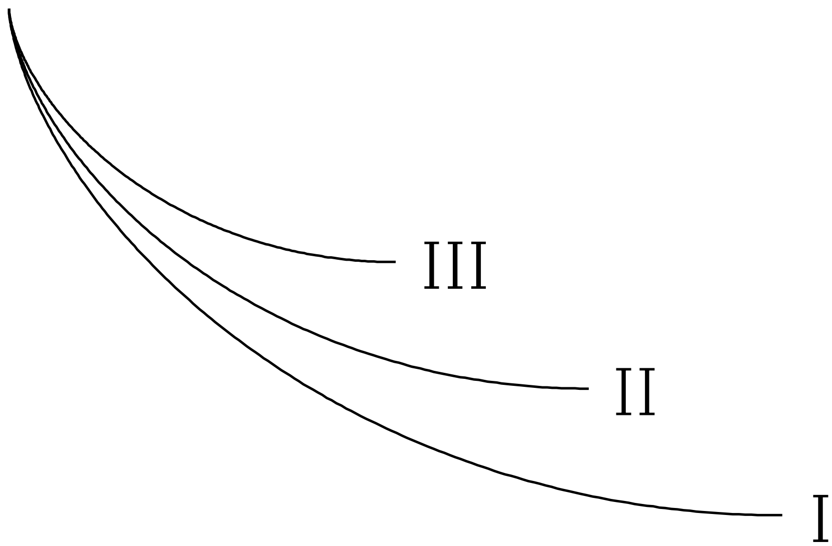


Contact: (Drew) milsom@email.arizona.edu

4: Let's now consider the normal force exerted by the track on the mass. Discuss how you expect its magnitude to change as the mass slides down the track. Explain your reasoning. Draw qualitatively correct free body diagrams at each of the six points in the figure. The first and last points are at the very top and the very bottom of the cycloid trajectory. Don't forget that the cycloid is perfectly vertical at the top and perfectly horizontal at the bottom. Hint: three different factors influence its magnitude.



5: Let's now consider three different brachistochrones with different "heights". The height ratios are 1.0 : 0.75 : 0.5. In each case, an identical mass is released from rest at the very top and slides to the bottom. When they reach the bottom, rank the size of the normal force in the three cases. Explain your reasoning. If you are not able to rank them, explain why you are having difficulty.



Comments about this tutorial:

While this was formally written during the Fall of 2019, some of these questions had been used for twenty years by the author. It can be used in a freshman mechanics class or a junior level classical mechanics course (prior to solving the problem quantitatively). Additionally, graduate students also make mistakes here.

Many students are going to need hints to perform well on these questions. While some of the hints could be added to the questions, I would rather learning assistants/instructors determine what hints to provide each student group.

We suggest that introductory students work in groups of four. Juniors enrolled in Lagrangian mechanics have not performed well on this when working individually. We suggest that they work in pairs.

To get through all this, they really need to be given 50 minutes.

Some students have spent far too much time on the first question (like 15 minutes) and then don't get very far. Having sufficient learning assistants to help them along would certainly be beneficial.

Comments about individual questions:

Question #1: One of the more common mistakes is to try to use energy conservation to answer this. The quick version is "the one that starts higher has more potential energy so it ends up moving faster so it can get there in the same time". One could draw a perfectly straight track and ask them why their argument doesn't work for that and see if that helps them recognize that the local track angle really matters.

Questions #2 & 3: There are lots of useful hints to provide here since most students really don't go about this in a logical way:

- Can you identify x (or v_x or etc.) at $t = 0$? Can you identify that quantity at the very bottom?
- What shape do you think that graph has between your endpoints and why? Does it increase continuously? Does it increase then decrease? Etc.
- Do you think Newton's Laws could help you draw these graphs? Why or why not?
- Do you remember how these graphs are related to each other? Do the ones you have drawn obey those relationships?
- Did you remember that y is defined so that it increases as the mass moves down?

Question #4: Our students have lots of practice drawing *qualitatively* correct free body diagrams. Nevertheless many students weren't even being careful to draw mg the same size so we had to remind many of them that we wanted to be able to compare the diagrams. Some definitely had explanations telling us that n had to get larger but their diagrams didn't show that.

Otherwise, most students won't understand that the acceleration perpendicular to the curve AND the relative orientation of $\vec{n}$ and $m\vec{g}$ both affect n. So general hints like "Do you think there is anything else that could influence n?" are useful.

The vast majority of the students don't consider the curving of the path so hints related to that are crucial. For example, "Is there anything about this that is similar to circular motion?"

Question #5: This question is primarily to get them to think. I would be thrilled if someone said "If the brachistochrone is larger then the mass will be moving faster at the bottom so that means the acceleration perpendicular to the curve must be larger so n should be larger. However, the brachistochrone will also be even flatter which should make that acceleration smaller. These two things at least partially cancel each other so I am not sure."

Tutorial source(s):

All questions were written by Drew Milsom. Clearly, every Lagrangian mechanics student solves the brachistochrone but most probably don't think about these issues at all.