Electromagnetic Waves Propagating in a Conductor

Firstname _______________ Lastname __________________

During this tutorial, you are going investigate some of the features of electromagnetic waves propagating in a conductor.

Part I

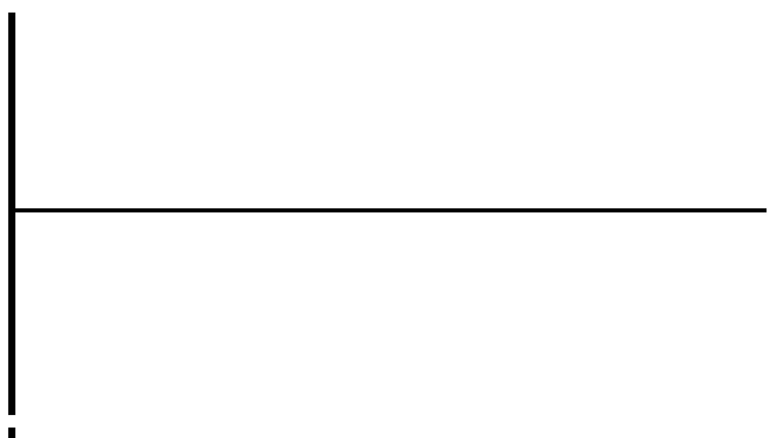
There is a continuously generated infinite plane wave with angular frequency ω propagating in the z direction in a linear isotropic and homogeneous conducting medium described by σ, μ, ϵ, $\rho_f = 0$ and $\vec{J}_f' = 0$. At this time, we are not letting the wave pass from a nonconductive medium into a conductive one since that would involve the possibility of reflection. We are only focusing on the propagation of a wave already inside a conducting medium.

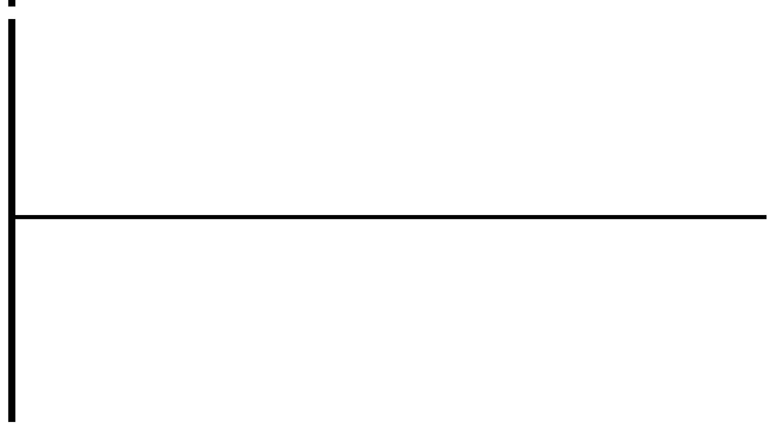
We have previously determined that *each* component of the electric and magnetic field satisfy the following scalar equation:

$$\left(\nabla^2 - \mu\sigma\frac{\partial}{\partial t} - \mu\epsilon\frac{\partial^2}{\partial t^2}\right) f(z,t) = 0 \tag{1}$$

and we have solved this when $\sigma = 0$. We now need to solve this including the term with σ.

1: Before attempting to solve this, let's think about what the solution should look like. On the top graph, graph f versus z at time t AND at time $t + \frac{\pi}{\omega}$. On the bottom graph, graph f versus t at z_1 and at $z_1 + \frac{\pi}{2k}$. Explain your graphs.





2: Now use the graphs you have drawn and guess a functional form for the solution for $f(z,t)$ and explain your guess. As a reminder, our solution in the non-conducting case was $f(z,t) = f_0 e^{i(kz-\omega t)}$. Could the solution be identical? Are there some differences? Explain.

3: Two students have answered Question #2 and they have obtained the following answers:

$$f_1(z,t) = f_0 e^{i(kz-\omega t)} e^{-\gamma z}, \qquad f_2(z,t) = f_0 e^{i(kz-\omega t)} e^{-\eta t}$$

where γ has units of m^{-1} and η has units of s^{-1}. For each possible solution, explain why it is correct or why it is incorrect.

4: Let $f(z,t) = f_0 e^{i(kz-\omega t)}$ and substitute it into Equation (1). Solve the resulting expression for k^2 to obtain the dispersion relation (the relationship between ω and k) for this system. Note that you expect k to be a complex number in this case.

5: Let $k = \alpha + i\beta$ so that the solution is really $f(z,t) = f_0 e^{i(\alpha z - \omega t)} e^{-\beta z}$. The $\text{Re}(k) = \alpha$ describes the oscillation and $\text{Im}(k) = \beta$ describes the damping. Solve the dispersion relation from Question #4 for α and β in terms of ω, μ, ϵ and σ.

In terms of a dimensionless parameter $Q = \dfrac{\omega\epsilon}{\sigma}$, the correct expressions are:

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\frac{1}{Q^2}}+1\right]^{1/2} \quad \text{and} \quad \beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\frac{1}{Q^2}}-1\right]^{1/2} \tag{2}$$

6: Verify that α and β in Equation (2) give the expected result in the limit that $\sigma \to 0$.

7: In Equation (2), is $\beta > 0$ or is $\beta < 0$? Is that consistent with what you expect? Explain.

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8: Let's examine the dimensionless parameter Q that I defined $\left(Q = \dfrac{\omega\epsilon}{\sigma}\right)$. Let's determine how to interpret this. Assume that $E \propto E_0 e^{-i\omega t}$. Determine $\frac{|\vec{J}_D|}{|\vec{J}_f|}$. Use that result to describe in words how to interpret Q and then determine if $Q \gg 1$ corresponds to a good insulator or a good conductor. Explain.

9: You could write $k = |k|e^{i\Omega}$. Determine the possible range of values for Ω and identify specific values for extremely good conductors and extremely good insulators.

10: Determine if the phase velocity of this wave is $\leq v_{\sigma=0}$ or $\geq v_{\sigma=0}$. Recall that the phase velocity describes the rate of change of the argument of the complex exponential so in this situation $v = v_{\text{phase}} = \frac{\omega}{\alpha}$.

11: Determine if the wavelength of the wave is $\leq \lambda_{\sigma=0}$ or $\geq \lambda_{\sigma=0}$.

Part II

Let's now determine if there is anything else different about the electromagnetic wave propagating through the conductor.

While it may or may not be correct, let's assume that we can write $\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$ and $\vec{B} = \vec{B}_0 e^{i(kz-\omega t)}$ where $k = \alpha + i\beta = k = |k|e^{i\Omega}$, $\vec{E}_0 = E_{0x}\hat{x} + E_{0y}\hat{y} + E_{0z}\hat{z}$ and $\vec{B}_0 = B_{0x}\hat{x} + B_{0y}\hat{y} + B_{0z}\hat{z}$. The E_{0x}, B_{0x}, etc. are amplitudes with no spatial/time dependence. In other words, we are still assuming that we have an incoming plane wave but we don't know (as of yet) the directions of $\vec{E}$ and $\vec{B}$. As we substitute these expressions into Maxwell's equations, we will determine whether there is anything different in this situation.

12: Substitute the electric field expression into Gauss' Law for electric fields. What do you learn from this? Is/are there any difference(s) from the non-conducting situation?

13: Substitute the magnetic field expression into Gauss' Law for magnetic fields. What do you learn from this? Is/are there any difference(s) from the non-conducting situation?

You should have obtained the same results as in the non-conducting case. The electric and magnetic fields are both transverse. Since we now know that, let's simplify our expressions. We can arrange our coordinate system so that the electric field in the wave lies along the x-axis so $\vec{E} = E_0 e^{i(kz-\omega t)}\hat{x}$. Since we are still somewhat unsure about the magnetic field direction, we will leave it as $\vec{B} = \vec{B}_0 e^{i(kz-\omega t)}$ with the restriction that we now know that there is no z-component.

14: Substitute the field expressions into Faraday's Law. What do you learn from this? Is/are there any difference(s) from the non-conducting situation?

You should have determined that $B_{0x} = 0$ so $\vec{E}$ and $\vec{B}$ are perpendicular, that their cross product determines the direction of propagation and that $E_0 k = \omega B_{0y} = \omega B_0$. You must be careful interpreting this last expression since k is a complex number so this actually means that $B_0 = \dfrac{E_0|k|e^{i\Omega}}{\omega}$. So while it may look the same at first glance, there is actually a phase difference between E_0 and B_0. **The electric and magnetic fields are no longer in phase!** As a reminder, you determined in Part I of this tutorial that $0 \leq \Omega \leq \frac{\pi}{4}$.

15: Since we now know that $\vec{E}$ and $\vec{B}$ and the wave propagation direction are mutually perpendicular, we can start with $E_0 e^{i(kz-\omega t)}\hat{x}$ and $B_0 e^{i(kz-\omega t)}\hat{y}$. Since there is a phase difference between $\vec{E}$ and $\vec{B}$, that just implies that E_0 and B_0 are complex numbers. Substitute these into Ampere's Law. What do you learn from this? Is/are there any difference(s) from the non-conducting situation?

You should have obtained $-ikB_0 = \mu\sigma E_0 - i\omega\mu\epsilon E_0$ which I will relabel $-ikB_0 = E_0(A_1 - A_2)$.

For the remainder of this tutorial, let's now focus specifically on the perfect conductor case ($\sigma \to \infty$).

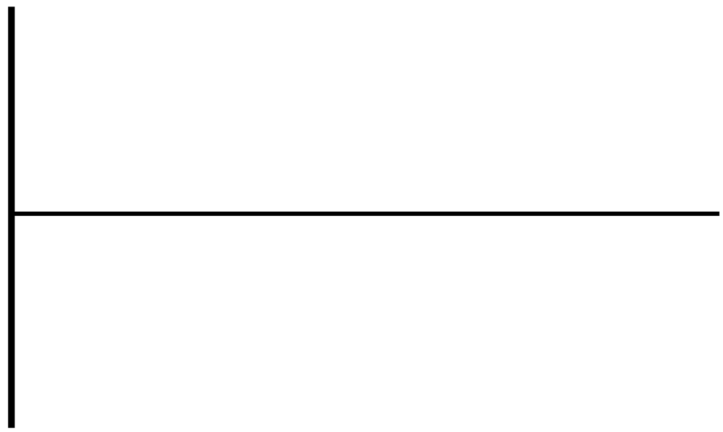
16: Which is larger A_1 or A_2? Explain. Hint: You may want to examine how those terms compare to the parameter Q that we previously defined.

17: Eliminate the smaller term and show that the relationship between B_0 and E_0 in Ampere's Law is exactly the same as you obtained in Question #14. What is the numerical value of the phase difference between $\vec{E}$ and $\vec{B}$?

18: Assume that you are at a particular z watching the electromagnetic wave pass by. Which peaks earlier in time: the electric field or the magnetic field? Explain.

19: The work you have just done illustrates that the magnetic field (in the perfect conducting case) is delayed by a phase of $\frac{\pi}{4}$ rad. Let's try and understand the source of this a bit more. We have: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{E}}{\partial t}$ and $\vec{\nabla} \times \vec{B} = \mu \vec{J}_f$ since the displacement current is negligible here. Faraday's Law *appears to say* that spatial derivatives of $\vec{E}$ and time derivatives of $\vec{B}$ are in phase so this would suggest that the fields themselves would be in phase. However, Ampere's Law says something **very** different. It says that spatial derivatives of $\vec{B}$ are in phase with $\vec{E}$. Considering that $\vec{E}$ and its time derivative differ by a phase of $\frac{\pi}{2}$ rad, the two equations **appear** to have different solutions.

Let's compare the Ampere's Law situation to a mechanics problem. Specifically, a mass m which is initially at rest is subjected to a force $F(t) = F_0 \cos \omega t$. Graph $F(t)$ and $v(t)$ on the same graph on the next page and make sure they are qualitatively correct. Explain your graph and/or calculate as needed. Identify which terms in the mechanics problem correspond to which terms in Ampere's Law.



So Faraday's Law and Ampere's Law *appear* to require different solutions. The former would like no phase shift between $\vec{E}$ and $\vec{B}$ while the latter would like it to be $\frac{\pi}{2}$ rad. **The final solution in the perfect conductor limit is the average of those phases!** This final solution satisfies both equations since k is complex. For example, when you take a spatial derivative of $\vec{E}$ you obtain $ik\vec{E}$ which is a term that can be written as $|k|e^{i(\frac{\pi}{2}+\Omega)}$. So the phase shift obtained when you take the derivative isn't just from the i, it is also influenced by k. Additionally, of course, the complex k also helps conserve energy since the field energy must drop as energy is transferred to the conduction electrons.

Other things to consider:

Is the Poynting vector different than in the non-conducting case?

Do both fields still have the same energy density?

How far does the wave have to propagate before the electric field is reduced in amplitude by a factor of e?

Comments about this tutorial:

This was written and used during the spring of 2017. It is VERY long. There were three tables of three students and one table of four students. The objective was to complete Part I in 75 minutes.

Assumed knowledge: They should have seen Equation #1 derived and seen its solution in the non-conducting case.

Paragraph before Question #1: The $\vec{J}'_f$ is the label used in the 2nd Edition of Electromagnetic Fields by Roald K. Wangsness. It distinguishes the current you directly associate with the conductivity from other free currents in the problem.

Questions #1-3: As expected, almost everyone drew the graphs incorrectly. Only two students initially had the amplitude decaying with position. Everyone else had a constant amplitude.

Most of those students ended up having the amplitude decaying in time but not all of them.

A couple students had difficulty understanding what $\frac{\pi}{2k}$ did to the wave. Even if they recognized that it shifted the wave by one-fourth of a cycle, they could not easily determine which way it shifted.

Although not written in the question itself initially (see below), I sketched a large conductor on the board and told them there was an infinite plane source that continously generated waves.

When everyone was having trouble drawing the graphs, I told them to read Questions #2 and #3 to see if that helped them analyze things.

I think by the end 80% of the students understood why the decay was in space but it did take a while. At each table, these three questions took at least 20 minutes and at the slowest table it was 30 minutes. I think this was definitely a very good exercise.

Question #4: Fine but there were some sign errors.

Question #5: Probably one-third of the students didn't realize that they could focus on the real and imaginary parts separately. Also there was some confusion about the difference between k^2 and k^*k. A few students did finish this but at three-fourths of the tables I had them just jump ahead and use the given answers.

Questions #6-7: Fine.

Question #8: A couple students didn't realize what $\vec{J}_D$ was. Another student was confused by whether $\vec{J}_D$ is displacement current or displacement current density - I think the language here is inconsistent as used by textbook authors and faculty.

Question #9: This created a lot of difficulty. Very few students realized that they need to analyze α and β in Equation #2 to determine the range. They quickly gave answers like $0 \rightarrow 2\pi$. I definitely had to get them started.

Question #10: Fine.

Question #11: Basically OK but there was some confusion over whether $\lambda = \frac{2\pi}{\alpha}$ or $\lambda = \frac{2\pi}{k}$.

At the end of 75 minutes, students at one table were just starting Question #10. The others finished and at two tables, they had started working on Question #12 in the last few minutes.

The remaining questions have not (as of yet) been used in a group activity setting. However, following Part I, I had the students work out Questions #12 & #14 before the next class. I then had them work Questions #15 & #19 during class as we discussed the rest of that topic.

Question #19: Some of you may not like this discussion since I begin by saying that Faraday's Law states that spatial derivatives of $\vec{E}$ and time derivatives of $\vec{B}$ are in phase but then later I point out that they really are not since k is complex. I have talked about this issue many times in class and it has always seemed beneficial to make the connection to driving forces in classical mechanics.

Changes made summer 2017:

Introduction to Part I: I added the phrase that there was continuous generation of these plane waves.

Tutorial source(s):

All questions were written by Drew Milsom.