Mathematical Necessities for Physics 371

1) Complex numbers

Let $z = x + iy$ be any complex number, where x and y are real, and $i = \sqrt{-1}$. Then $x = \mathrm{Re}z$ and $y = \mathrm{Im}z$. The *complex conjugate* of z is defined to be $z^* = x - iy$. Note that

$$zz^* = (x+iy)(x-iy) = x^2 + y^2 \equiv |z|^2.$$

Complex numbers can also be expressed in polar coordinates using the identity $e^{i\theta} = \cos\theta + i\sin\theta$. Let $z = re^{i\theta}$, where r and θ are real. Then $r = |z| = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$.

Examples: i) If $z_1 = z_2$ then $x_1 = x_2$ *and* $y_1 = y_2$. ii) $z/z^* = \frac{x+iy}{x-iy} = e^{i2\theta}$.

For the purposes of this course, algebra and calculus with complex numbers is entirely analogous to that with real numbers.

2) Gaussian integrals

Define $I(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx$.

i) $I(\alpha) = \sqrt{\pi/\alpha}$.

ii) $-dI/d\alpha = \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx$.

iii) Clearly, $(-1)^n d^n I/d\alpha^n = \int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx$.

3) Dirac delta function

Define

$$\delta(x - x_0) \equiv \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x_0) - \epsilon k^2}.$$

The delta function has the important property

$$\int_a^b f(x)\delta(x-c)dx = \begin{cases} f(c) & \text{if } a < c < b, \\ 0 & \text{otherwise.} \end{cases}$$

4) Fourier transforms

Let

$$\tilde{\psi}(k) = \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx.$$

Then

$$\psi(x) = \int_{-\infty}^{\infty} \tilde{\psi}(k) e^{ikx} \frac{dk}{2\pi}.$$

5) Probability

Let $\rho(x)$ be the probability density in one dimension. The probability to find a particle in the interval (a, b) is $P(a, b) = \int_a^b \rho(x)dx$. The total probability to find the particle *somewhere* should be unity: $\int_{-\infty}^{\infty} \rho(x)dx = 1$. The average value of any function $f(x)$ is

$$\langle f \rangle = \int_{-\infty}^{\infty} f(x)\rho(x)dx.$$

The *variance* of a function is defined to be

$$\text{var} f = \delta f^2 \equiv \langle (f - \langle f \rangle)^2 \rangle = \langle f^2 \rangle - \langle f \rangle^2 \geq 0.$$

The *standard deviation* $\delta f \equiv \sqrt{\text{var} f}$. δf is a measure of the range of likely values of f.

These formulas have nothing particular to do with quantum mechanics; this is just probability theory.

6) Linear algebra

The *transpose* of a matrix A is denoted by A^{T}. If A_{ij} is the ijth matrix element of A, then $\left(A^{\text{T}}\right)_{ij} = A_{ji}$.

The *Hermitian conjugate* of a matrix A is denoted by $A^\dagger$. Note that $A^\dagger = (A^*)^{\text{T}}$.

If the matrix A and the vector $\mathbf{a}$ satisfy the equation

$$A\mathbf{a} = \lambda \mathbf{a},$$

where λ is a number, then $\mathbf{a}$ is said to be an *eigenvector* of A, and λ is the corresponding *eigenvalue*.

The eigenvalues of a square matrix A are determined by the characteristic equation

$$\det\{A - \lambda I\} = 0,$$

where I is the unit matrix.

A *hermitian* matrix H satisfies $H^\dagger = H$. The eigenvalues of such a matrix are real, and the eigenvectors can be chosen orthonormal.

A *unitary* matrix U satisfies $U^\dagger = U^{-1}$. The eigenvalues of such a matrix lie on the unit circle in the complex plane, and the eigenvectors can be chosen orthonormal. Every unitary matrix can be related to a corresponding hermitian matrix: $U = e^{iH}$.