Exercises for Physics 325

Problem Set 1; Due January 28, 1999

1) Birthday problem

There are 21 students in this class. Assume that each student is equally likely to have been born on any given day (neglect leap year).

a) What is the probability that at least one other student has the same birthday as you do?

b) What is the probability that at least 2 students have the same birthday? Hint: $\ln n! \approx n \ln n - n + \ln \sqrt{2\pi n}$.

2) Drunken diffusion

A drunkard in a small town with only a single street stumbles out of a pub at closing hour. Estimate how long it will take him to reach his apartment 100 meters down the street if he takes steps of length 1 meter once per second, but is so drunk that he steps forward or backward with equal probability. (Consider his motion to be one-dimensional.)

3) Poisson distribution I

The probability $P(n)$ that an event characterized by a probability p occurs n times in N trials is given by the binomial distribution

$$P(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}.$$

Consider a situation where the probability $p \ll 1$ and $n \ll N$. Several approximations can then be made to reduce $P(n)$ to a simpler form.

a) Using the result $\ln(1-p) \approx -p$, show that $(1-p)^{N-n} \approx e^{-Np}$.

b) Show that $N!/(N-n)! \approx N^n$.

c) Hence show that

$$P(n) \approx \frac{\lambda^n}{n!} e^{-\lambda},$$

where $\lambda = Np$. This is referred to as the "Poisson distribution."

4) Poisson distribution II

a) Show that the Poisson distribution is properly normalized, i.e.,

$$\sum_{n=0}^{N} P(n) = 1.$$

(The sum can be extended to infinity to an excellent approximation since $P(n)$ is negligibly small when $n > N$.)

b) Use the Poisson distribution to calculate $\langle n \rangle$.

c) Use the Poisson distribution to calculate the variance

$$\operatorname{var} n = \langle (n - \langle n \rangle)^2 \rangle.$$

5) Consider a gas of N_0 noninteracting molecules enclosed in a container of volume V_0. Focus attention on any subvolume V of this container and denote by N the number of molecules located within this subvolume. Each molecule is equally likely to be located anywhere within the container; hence the probability that a given molecule is located within the subvolume V is simply equal to V/V_0.

a) What is the mean number $\langle N \rangle$ of molecules located within V? Express your answer in terms of N_0, V_0, and V.

b) Find the relative dispersion $\operatorname{var} N/\langle N \rangle^2$. Express your answer in terms of $\langle N \rangle$, V and V_0.

c) What does the answer to part (b) become when $V \ll V_0$?

d) What value should the dispersion assume when $V \to V_0$? Does the answer to part (b) agree with this expectation?

6) Suppose that in the preceding problem the volume V under consideration is such that $V/V_0 \ll 1$ but $\langle N \rangle \gg 1$. What is the probability that the number of molecules in this volume is between N and $N + dN$?

7) Useful integrals

Evaluate the following integrals:

$$I_1(n) = \int_0^\infty dx\, e^{-x} x^n$$

$$I_2(n) = \int_{-\infty}^\infty dx\, e^{-\alpha x^2} x^n$$

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Problem Set 2; Due February 11, 1999

Kittel & Kroemer: Problems 2.3 and 2.5

Reif: Problem 2.7

4) Consider a system of N localized weakly-interacting spin-1/2 particles of magnetic moment μ in an external magnetic field B. The quantum energy levels for a single particle are $E = \pm\mu B$. The total energy of the system is $E = -(N_\uparrow - N_\downarrow)\mu B$, and the total number of configurations with a given energy is

$$\Omega(m) = \frac{N!}{[(N+m)/2]![(N-m)/2]!},$$

where $m = N_\uparrow - N_\downarrow$, as shown in the lecture.

a) Using the definition of temperature

$$\frac{1}{T} = \frac{\partial \ln \Omega}{\partial E}$$

and Stirling's approximation for the factorial, obtain a relation between the energy of the system and the temperature, valid for arbitrary temperatures.

b) Under what circumstances is T negative?

c) What is the total magnetic moment M of the system?

5) Consider two spin systems like that in the previous problem with $N_1 = N_2 = 10^{20}$. Assume that system 1 is intially at temperature $T = 0$ and system 2 is initially at temperature $T = \mu B$. The systems are then brought into thermal contact and allowed to reach equilibrium.

a) Calculate the equilibrium temperature.

b) Calculate the probability to find system 1 again in its initial state of minimum energy after equilibrium has been established. Hint: Use the Gaussian approximation to estimate the total number of states accessible to the combined system in equilibrium.

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Problem Set 5; Due March 11, 1999

1) In the lectures, it was shown that the equilibrium thermal noise voltage $V(t)$ across a resistor of resistance R satisfies

$$\langle V^2 \rangle = \int_0^\infty J_+(\omega)\, d\omega,$$

with the following *power spectrum*,

$$J_+(\omega) = \frac{2R}{\pi} \left(\frac{\hbar\omega}{\exp(\hbar\omega/T) - 1} \right).$$

For low frequencies or large temperatures, the power spectrum simplifies to the classical result

$$J_+(\omega) \simeq \frac{2RT}{\pi}. \tag{1}$$

Calculate the frequency $\nu_{1/2} = \omega_{1/2}/2\pi$ in Hz at which the power spectrum is reduced by 50% from the classical result (1) at room temperature. What do you conclude about the validity of the classical approximation (1) in most electronics applications?

2–6) Kittel & Kroemer: Problems 5.1, 5.3, 5.4, 5.6, and 5.8

Exercises for Physics 325

Problem Set 10; Due Wednesday, April 28, 1999

1–3) Kittel & Kroemer: Problems 10.1, 10.2, and 10.3

4) Mean-field theory of a ferromagnet

For a lattice of spin-1/2 atoms interacting via the Heisenberg-Ising exchange interaction

$$E_{ij} = \begin{cases} -JS_i^z S_j^z, & i, j = \text{nearest neighbors}, \\ 0, & \text{otherwise}, \end{cases}$$

the following self-consistent mean-field equations were derived in the lecture for the thermal average spin $\langle S^z \rangle$ of each atom (a factor of 2 error has been corrected):

$$\langle S^z \rangle = \frac{\hbar}{2} \tanh\left(\frac{g\mu_B \hbar B_{\text{eff}}}{2T} \right), \tag{2}$$

$$B_{\text{eff}} = B + \frac{nJ}{g\mu_B} \langle S^z \rangle, \tag{3}$$

where n is the number of nearest neighbors.

a) Calculate the critical temperature T_c above which the only solution of Eqs. (1) and (2) for $B = 0$ is $\langle S^z \rangle = 0$.

b) The total magnetic moment of the system is $M = Ng\mu_B \langle S^z \rangle$. Calculate the magnetic susceptibility

$$\chi(T) = \left. \frac{\partial M}{\partial B} \right|_{B=0}$$

for $T > T_c$.

c) Comment on the behavior of $\chi(T)$ as $T \to T_c$.