**Exercises for Physics 562**

Problem Set 1; September 4, 1998

1) 1D scattering matrix

Consider a particle moving in the following one-dimensional potential:

The potential is $V(x) = V\theta(x) + \nu\delta(x)$, where

$$\theta(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

and $\delta(x)$ is the Dirac delta function.

a. Consider a particle incident from the left ($b = 0$). Find the amplitudes $a'$ and $b'$ of the reflected and transmitted waves by solving the Schrödinger equation.

b. Repeat the calculation for a particle incident from the right ($a = 0$).

c. Calculate the currents flowing on each side of the barrier for each case, and verify that they are equal.

d. Calculate the transmission probability for each case.

e. The scattering matrix $\mathbf{s}$ relates the incident and scattered current amplitudes:

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \mathbf{s} \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

where $\alpha = \sqrt{v_1}\, a$, $\alpha' = \sqrt{v_1}\, a'$, $\beta = \sqrt{v_2}\, b$, and $\beta' = \sqrt{v_2}\, b'$ are the amplitudes of the currents $j_1 = |\alpha|^2 - |\alpha'|^2$ and $j_2 = |\beta'|^2 - |\beta|^2$ flowing on the left and right sides of the barrier, respectively. Calculate $\mathbf{s}$ for the above potential, and verify that $\mathbf{s}$ is unitary and symmetric.

f. Show that current conservation requires that $\mathbf{s}$ be unitary for an arbitrary potential. Show that $\mathbf{s}$ is symmetric if the Schrödinger equation is time-reversal invariant.

2) Transfer matrix

The transfer matrix for a one-dimensional system is defined by

$$\begin{pmatrix} b' \\ b \end{pmatrix} = \mathbf{t} \begin{pmatrix} a \\ a' \end{pmatrix}.$$

a. Calculate $\mathbf{t}$ for the above potential.

b. Verify that

$$\mathbf{t} = \sqrt{v_1/v_2} \begin{pmatrix} 1/t^* & -r^*/t^* \\ -r/t & 1/t \end{pmatrix},$$

where $r$ and $t$ are the elements of the matrix $\mathbf{s}$, and prove that this relation is valid for an arbitrary potential $V(x)$.

**Exercises for Physics 562**

Problem Set 2; September 18, 1998

## 1D Resonant Tunneling

Consider the scattering of a particle with wave-vector $k$ and energy $E = \hbar^2 k^2/2m$ from the double-barrier potential sketched above. Assume that the barriers can be described by energy-independent scattering matrices:

$$\mathbf{s_j} = \begin{pmatrix} -iR_j^{1/2} & T_j^{1/2} \\ T_j^{1/2} & -iR_j^{1/2} \end{pmatrix}; \quad j = 1, 2.$$

a. Calculate the scattering matrix for the entire system from the transfer matrices of barriers 1 and 2 and the transfer matrix of the region between the barriers, where the particle acquires a phase $\phi = kd$.

b. Find the transmission amplitude through the combined system by summing over all possible scattering trajectories à la Feynman.

c. Give the energies $E_n$ of the bound states in the potential well for the case where the barriers are impenetrable.

d. Find the partial escape rates $\Gamma_{1n}/\hbar$ and $\Gamma_{2n}/\hbar$ of a particle in the state $E_n$ through the two barriers for the case where the other barrier remains impenetrable (hint: consider the motion between the two barriers classically).

e. In the limit of nearly impenetrable barriers, show that the transmission probability of the double-barrier takes the Breit-Wigner form

$$T(E) \simeq \frac{\Gamma_{1n}\Gamma_{2n}}{(E - \tilde{E}_n)^2 + (\Gamma_{1n} + \Gamma_{2n})^2/4}$$

in the vicinity of the energies of the bound states.

**Exercises for Physics 562**

Problem Set 3; October 2, 1998

**Aharonov-Bohm Effect**

1) Consider a wave-splitter which connects 4 one-dimensional conductors. Construct a scattering matrix such that an electron incident from conductor 1 is transmitted with probability 0.5 into conductor 3 and with probability 0.5 into conductor 4; likewise an electron incident from conductor 2 is transmitted with probability 0.5 into conductor 3 and with probability 0.5 into conductor 4. Such a wave-splitter also describes the transmission and reflection of light from a half-silvered mirror.

2) Use the wave-splitter determined above to construct an Aharonov-Bohm ring: Connect conductor 3 of the left wave-splitter to conductor 3 of the right wave-splitter to form the upper half of the ring, of length $\ell_1$, and connect conductor 4 of the left wave-splitter to conductor 4 of the right wave-splitter to form the lower half of the ring, of length $\ell_2$. Assume that there is a magnetic flux $\Phi$ through the ring. Calculate the probabilities of transmission $T_{11}$, $T_{12}$, $T_{21}$, and $T_{22}$ as a function of $\Phi$. Convince yourself that the reflection probabilities are zero.

3) Suppose that conductors 1 and 2 of the left wave-splitter are attached to an electron reservoir and that conductors 1 and 2 of the right wave-splitter are connected to another electron reservoir. Calculate the conductance measured between the two reservoirs at zero temperature. Show that the conductance is quantized and independent of $\Phi$!

4) Now consider an Aharonov-Bohm ring made by connecting conductor 4 of the left wave-splitter to conductor 4 of the right wave-splitter to form the upper half of the ring, of length $\ell_1$, and connecting conductor 2 of the left wave-splitter to conductor 2 of the right wave-splitter to form the lower half of the ring, of length $\ell_2$. Calculate the transmission probabilities $T_{11}$, $T_{13}$, $T_{31}$, and $T_{33}$ by summing over all possible scattering paths. What are the reflection probabilities?

5) Calculate the conductance of this ring at zero temperature when wires 1 and 3 of the left wave-splitter are connected to one reservoir, and wires 1 and 3 of the right wave-splitter are connected to another reservoir. Discuss the conductance as a function of $\Phi$, the dimensions of the ring, and the wave-vector of the incident electron. Why is the conductance of this ring $\Phi$-dependent, while that of the ring in problem 3 was independent of $\Phi$?

**Exercises for Physics 562**

Problem Set 4; October 23, 1998

## 1) Density of states

Show that the expression for the density of states given in the lecture

$$\frac{dN(E)}{dE} = \frac{1}{4\pi i}\mathrm{Tr}\left[S^\dagger \frac{\partial S}{\partial E} - S\frac{\partial S^\dagger}{\partial E}\right]$$

implies that the total number of states with energy $\leq E$ is

$$N(E) = \frac{1}{2\pi i}\ln\det S(E) + C,$$

where $C$ is a constant of integration. Hint: Consider a basis of eigenstates of $S(E)$.

## 2) Quasi-bound state

Consider the $M \times M$ scattering matrix for resonant tunneling derived in the lecture:

$$S_{mn} = -\delta_{mn} + \frac{i\sqrt{\Gamma_m \Gamma_n}}{E - E_r + i\Gamma/2},$$

where $\Gamma = \sum_{n=1}^{M}\Gamma_n$. Calculate $N(E)$. (Hint: Consider $M = 1, 2, 3, \ldots$) Show that $N(E)$ increases by one as $E$ increases from $-\infty$ to $\infty$. Discuss your result.