Homework problem: partial decomposition of the isospin breaking π production operator at threshold

Definition of momentum eigenstates

$|p(ls)jm>$ are eigenstates for the momentum p, orbital angular l and NN spin state s. l and s couple to the total angular momentum j with third component m.

The states are normalized according to

$$< p'(l's')j'm'|p(ls)jm >= \frac{\delta(p-p')}{p\ p'}\ \delta_{ll'}\delta_{ss'}\delta_{jj'}\delta_{mm'} \tag{1}$$

In 3D representation that reads

$$< \vec{p}'\ m_1 m_2|p(ls)jm >= \sum_{\mu}(lsj, \mu\ m-\mu)\ (lsj, m_1\ m_2\ m-\mu)\ Y_{l\mu}(\hat{p}')\ \frac{\delta(p-p')}{p\ p'} \tag{2}$$

The momentum is the conjugated momentum to the relative distance of the two nucleons $\vec{r}_1 - \vec{r}_2$. $\vec{k}_1$ and $\vec{k}_2$ are the individual momenta of two nucleons. Then $\vec{p} = 1/2\ (\vec{k}_1 - \vec{k}_2)$.

Problem: Calculate the spin-orbit part of the production operator

The spin-orbit part of the production operator reads

$$O_{12} = -\frac{f}{4\pi m_\pi F_\pi^2 \sqrt{2m_\pi}}\ (\vec{\sigma}(1) - \vec{\sigma}(2)) \cdot (-i\hat{r})\ \frac{d}{dr}\ \left(\frac{e^{-m_\pi r}}{r}\right) \tag{3}$$

This operator is defined in configuration space: $\vec{r} = r\ \hat{r} = \vec{r}_1 - \vec{r}_2$. $\vec{\sigma}(i)$ are the Pauli matrices acting on the spin of particle i.

Hopefully you can confirm

$$\begin{aligned}
&< p(ls)jm|O_{12}|p'(l's')j'm' > \\
&= \delta_{jj'}\delta_{mm'}\ \left(-\frac{f}{4\pi m_\pi F_\pi^2 \sqrt{2m_\pi}}\right)\ \frac{2}{\pi}\ \int dr r^2\ j_l(pr)\ \frac{d}{dr}\ \left(\frac{e^{-m_\pi r}}{r}\right) j_{l'}(p'r) \\
&(-6\sqrt{3})\ i^{l'-l+1}\ \sqrt{(2j'+1)(2l+1)(2s+1)(2s'+1)} \left\{ \begin{array}{ccc} j & j' & 0 \\ l & l' & 1 \\ s & s' & 1 \end{array} \right\} \\
&(l\ 1\ l', 00) \left\{ \begin{array}{ccc} s & s' & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \end{array} \right\}\ (1-(-1)^{s+s'})
\end{aligned} \tag{4}$$

Intermediate results

I use for the definition of the reduce matrix elements for integer rank tensor operators

$$< L\ M|O^k_\mu|L'\ M' >= (L'\ k\ L, M'\ \mu\ M) < L||O^k||L' > \tag{5}$$

Using this notation I get for the reduced matrix element of the spin operator

$$< s||\sigma(1) - \sigma(2)||s' >= 6\sqrt{(2s'+1)} \left\{ \begin{array}{ccc} s & s' & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 0 \end{array} \right\} (1 - (-)^{s+s'}) \tag{6}$$

The reduced matrix element of the orbital part reads

$$< pl||(-i\hat{r})\ \frac{d}{dr}\ \left(\frac{e^{-m_\pi r}}{r}\right) ||p'l' >= i\ (-i)^l\ i^{l'}\ \ (l\ 1\ l', 00)\ A_{ll'}(p,p') \tag{7}$$

The Fourier transform of the exponential is calculated numerically

$$A_{ll'}(p,p') = \frac{2}{\pi}\ \int_0^\infty drr^2\ j_l(pr)\ \frac{d}{dr}\ \left(\frac{e^{-m_\pi r}}{r}\right) j_{l'}(p'r) \tag{8}$$

Isospin part

The Pauli matrices for isospin part appear in two combinations: $\tau(1)\ \tau(2)$ and $\tau_3(1)\ \tau_3(2)$. I use states to fixed isospin t, but I distinguish pp,np and nn systems. The isospin is given by the antisymmetry requirement $(-1)^{l+s+t} = (-1)$. The isospin is well defined, because the spin orbital part of the operator does not change the symmetry of the subsystem as it changes the spin s and the angular momentum l by 1. I admit that this is strange mixture of isospin formalism and particle basis. I use that for the NN interaction, too. Here it has the advantage, that the operators can be easily written down. The first operator results in (-3) for $t = 0$ and 1 for $t = 1$ states. The second opertor is 1 for the nn and pp systems, but (-1) for the np system.