Huge Effect of Initial State Interactions- the "Weinberg term operator" can dominate the $\sigma(DD \to \alpha\pi^0)$ Reaction

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We show that including the effects of initial state interactions enhances the effect of the the Weinberg homework operator $\mathcal{O}$ from a cross section of 0 (for simple wave functions) to a large one. In particular, if just the OPEP is used for initial state interactions, one gets $\sigma \approx 12$ pb from $\mathcal{O}$ in an oversimplified calculation. This example shows that initial state interactions are very important.

The problem is defined by stating the initial and final wave functions and the interaction operator. For threshold production of a pion the initial state must be $J^\pi = 0^-$. The fully symmetrized initial wave function is

$$\begin{aligned} |I\rangle &= |\mathbf{k}, M_1, M_2\rangle \\ &= \frac{1}{\sqrt{6}}(1+P_{23}+P_{24})\left(e^{i\mathbf{k}\cdot(\mathbf{R_1}-\mathbf{R_2})} + (-1)^S e^{-i\mathbf{k}\cdot(\mathbf{R_1}-\mathbf{R_2})}\right)\sum_{S,M_s}|S,M_s\rangle\langle S,M_s|1M_1,1M_2\rangle \quad (1)\\ & |1M_1,1M_2\rangle \equiv |D_{1,2}(M_1)\otimes D_{3,4}(M_2)\rangle, \quad (2)\end{aligned}$$

where $S=1$ for the 0^- state, the notation $D_{1,2}(M_1)$ stands for a deuteron, made of nucleons 1 and 2, in a total spin state of M_1, and P_{23} is a particle interchange operator. For the homework problem we take the deuteron to be in an S-state only. It is actually better to use a different initial state for simple calculations:

$$\begin{aligned} |I\rangle_N &= |\mathbf{k}, M_1, M_2\rangle\rangle \\ &= e^{i\mathbf{k}\cdot(\mathbf{R_1}-\mathbf{R_2})}|1M_1,1M_2\rangle. \end{aligned} \quad (3)$$

This is legitimate if one uses a fully anti-symmetrized ^4He wave function and uses a transition operator that is symmetric under particle interchange.

The pion production operator is taken (at threshold, and neglecting terms of order m_π/m_N) for the homework problem as due to the quark mass difference effects[1]

$$\mathcal{O} \equiv \sum_{i<j}\mathcal{O}_{i,j}, \quad (4)$$

$$\begin{aligned}\mathcal{O}_{i,j} &= -\frac{f}{4\pi\mu F_\pi^2\sqrt{2\mu}}[\delta m_N(\boldsymbol{\tau}^{(i)}\cdot\boldsymbol{\tau}^{(j)} + \tau_3^{(i)}\tau_3^{(j)}) - \bar{\delta} m_N(\boldsymbol{\tau}^{(i)}\cdot\boldsymbol{\tau}^{(j)} - \tau_3^{(i)}\tau_3^{(j)})] \\ &\quad (\boldsymbol{\sigma}^{(i)} - \boldsymbol{\sigma}^{(j)})\cdot(-i\hat{\boldsymbol{r}})\frac{d}{dr}\left(\frac{e^{-\mu r}}{r}\right), \end{aligned} \quad (5)$$

$$\mathbf{r} \equiv \mathbf{r}_i - \mathbf{r}_j \quad (6)$$

where $F_\pi = 186$ MeV is the pion decay constant, μ is the pion mass, and f is the pion-nucleon coupling constant ($f = g_A\mu/F_\pi$ in leading order; $f^2/4\pi = 0.075$ is used here). The parameters, δm_N and $\bar{\delta} m_N$, account for the effect of the current quark mass difference

$m_d - m_u$ and the electromagnetic contribution. The sum of these is fixed by the nucleon mass difference $m_n - m_p = 1.3$ MeV. Here we use $\delta m_N = 3.3$ MeV and $\bar{\delta} m_N$= -2 MeV.

Note the appearance of the factor $\frac{1}{\sqrt{2\mu}}$. This is a phase space factor, valid only at threshold. Its appearance has a historical origin, rooted in models which treat the nucleon as static. It is better to leave this out of the interaction, and to use the relativistic phase space. Thus define

$$V \equiv \mathcal{O}\sqrt{2\mu}. \tag{7}$$

Such replacements should be made in all operators.

The final state wave function is given by

$$
\begin{aligned}
|F\rangle &= |\alpha\rangle \otimes |\pi^0\rangle & (8)\\
\langle \mathbf{r}_i, s_i, t_i|\alpha\rangle &= Ne^{-b\left(\xi_1^2+\xi_2^2+\xi_3^2\right)}|S,T\rangle & (9)\\
|S,T\rangle &= \frac{1}{\sqrt{2}}\left(|\phi'\rangle|\chi''\rangle - |\phi''\rangle|\chi'\rangle\right), & (10)
\end{aligned}
$$

where $|\phi'(\chi')\rangle$ is the spin (isospin) wave function for particles (1,2) coupled to zero and (3,4) coupled to zero; and, $|\phi''(\chi'')\rangle$ is the spin (isospin) wave function for particles (1,2) coupled to unity and (3,4) coupled to unity:

$$
\begin{aligned}
|\phi'\rangle &= |S=0\rangle_{1,2} \otimes |S=0\rangle_{3,4} & (11)\\
|\phi''\rangle &= \frac{1}{\sqrt{3}}\left[|11\rangle_{1,2}\otimes|1-1\rangle_{3,4} - |1-1\rangle_{1,2}\otimes|11\rangle_{3,4} - |10\rangle_{1,2}\otimes|10\rangle_{3,4}\right]. & (12)
\end{aligned}
$$

The value of b is chosen to yield the experimental vale of the rms radius, which is 1.67 fm.

The spatial Jacobi variables are given by

$$
\begin{aligned}
\xi_1 &= \frac{\mathbf{r}_1 - \mathbf{r}_2}{\sqrt{2}} & (13)\\
\xi_2 &= \frac{\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3}{\sqrt{6}} & (14)\\
\xi_3 &= \frac{\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 - 3\mathbf{r}_4}{\sqrt{12}} & (15)\\
\xi_4 &= \frac{\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4}{4}. & (16)
\end{aligned}
$$

A very useful set of variables is given by

$$
\begin{aligned}
\mathbf{r} &= \left(\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_3 - \mathbf{r}_4\right)/2 & (17)\\
\rho_1 &= \mathbf{r}_1 - \mathbf{r}_2 & (18)\\
\rho_2 &= \mathbf{r}_3 - \mathbf{r}_4. & (19)
\end{aligned}
$$

Then the ${}^4He$ spatial wave function is given by

$$\Psi_{He} = \left[\frac{64}{\pi^{9/2}\alpha^9}\right] \exp\left[-(2r^2 + \rho_1^2 + \rho_2^2)/\alpha^2\right], \tag{20}$$

with $\alpha = 2.77$ fm. This is the same as the wave function given above. Chuck and Anders also suggest the simplified deuteron wave function:

$$\Phi_D(\rho) = \left[\frac{1}{\pi^{3/2}\beta^3}\right]^{1/2} \exp\left[-(\rho^2/2\beta^2)\right], \tag{21}$$

with $\beta = 3.21$ fm. The initial and final wave functions and the interaction operator for the homework problem have been specified. The goal is to calculate the threshold cross section. The CSB collaboration has found that

$$\langle F|V|I\rangle = 0. \tag{22}$$

Here we examine the role of initial state interactions arising from the one pion exchange operator.

Effect of Distortions We have the pion-production operator

$$V = \sum_{i<j} V_{i,j}, \tag{23}$$

recall eq. (7). To get a first estimate of the effects of distortion we replace this by

$$T \equiv \sum_{i<j} V_{i,j} G_0(i,j) V_{i,j}^{OPEP} \equiv \sum_{i,j} T_{i,j}. \tag{24}$$

We shall take particles i, j to arise from different deuterons. Then each nucleon is assumed to have a momentum of $\pm\mathbf{k}/2$, schematically

$$G_0(i,j) = \frac{m_N}{\kappa^2 - p^2(i,j)}, \tag{25}$$

$$\kappa \equiv k/2, \tag{26}$$

where $p^2(i,j)$ is the square of the relative momentum operator for the pair i, j, and m_N is twice the reduced mass of the i, j NN system. The OPEP is given by

$$V_{i,j}^{OPEP} = \frac{g^2}{4\pi}\frac{\mu^3}{4m_n^2}\left[S_{i,j}\left(\frac{1}{(\mu r)^3} + \frac{1}{(\mu r)^2} + \frac{1}{3\mu r}\right)\exp\left(-\mu r\right) + \frac{\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}{3}\left(\frac{\exp\left(-\mu r\right)}{\mu r} - \frac{4\pi\delta(\mathbf{r})}{\mu^3}\right)\right], \tag{27}$$

where $\mathbf{r} = \mathbf{r}_i - \mathbf{r}_j$. The operator $S_{i,j}$ is the usual tensor operator. We shall **throw away** the delta function for our first estimate, but this must be looked at later.

The key effect occurs in the action of the spin and isospin terms of the OPEP upon the spin and isospin terms of the operator V. In prepartion let us note that

$$\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\ \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j = 3 - 2\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j \tag{28}$$

$$\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\ \boldsymbol{\tau}_i\cdot\mathbf{n}\ \boldsymbol{\tau}_j\cdot\mathbf{n} = 1 + \boldsymbol{\tau}_i\cdot\mathbf{n}\boldsymbol{\tau}_j\cdot\mathbf{n} - \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j, \tag{29}$$

where $\mathbf{n} \equiv \hat{z}$. Thus the terms quadratic in isospin get terms which do not involve isospin and can connect the deuteron to the $T = 0, T = 0$ component of the alpha particle. Note also that

$$(\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_j)\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j = -2i(\boldsymbol{\sigma}_i\times\boldsymbol{\sigma}_j)\cdot\hat{\mathbf{r}} - (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_j)\cdot\hat{\mathbf{r}}. \tag{30}$$

The tensor force does not contribute with our HW wave functions. This is because after the angular integrations, the matrix element is of the form of a rank-2 tensor in spin dotted with with rank-2 tnesor in the vector $\mathbf{k}$. The rank-2 spin tensor can not connect the spin one state of the deuteron with the required spin 0 pair in the ^{4}He wave function.

Then

$$T_{i,j} = \frac{f}{4\pi\mu F_\pi^2}\, 2\left[\delta m_N - \bar{\delta} m_N\right](-2)(\boldsymbol{\sigma}_i \times \boldsymbol{\sigma}_j)\cdot\hat{\boldsymbol{r}}\left(\frac{d}{dr}\frac{e^{-\mu r}}{r}\right)\psi(\boldsymbol{r}), \tag{31}$$

with

$$\psi(\boldsymbol{r}) = \int d^3r' G_0(\boldsymbol{r},\boldsymbol{r}')V_0(\boldsymbol{r}')e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}'} \tag{32}$$

$$G_0(\boldsymbol{r},\boldsymbol{r}') \equiv m_N \frac{e^{i\kappa|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{33}$$

$$V_0(r') \equiv \frac{g^2}{4\pi}\frac{\mu^2}{12m_N^2}\frac{e^{-\mu r'}}{r'}. \tag{34}$$

Note the factor $2\left[\delta m_N - \bar{\delta} m_N\right]$ which arises from combing both terms of $\mathcal{O}$ and using Eqs. (28,29). The two terms have the same sign and a factor of 2.

To get a first estimate we replace $\psi(\boldsymbol{r})$ by $\psi(\boldsymbol{r}=\mathbf{0})$. Then one can find that

$$\psi(\boldsymbol{r}=\mathbf{0}) = \frac{m_N}{k}\frac{g^2}{4\pi}\frac{\mu^2}{12m_N^2}\left[\frac{i}{2}\ln(1+\frac{k^2}{\mu^2}) + (\frac{\pi}{2} - \tan^{-1}\frac{\mu}{k})\right] \tag{35}$$

$$\approx 0.065(1+i), \tag{36}$$

where the number is obtained with our kinematics. We see that the magnitude of $\psi(0)$ is about 0.1 which is greater than $\frac{\mu}{2m_N}$.

Given the operator $T_{i,j}^{(0)}$:

$$T_{i,j}^{(0)} \equiv \frac{f}{4\pi\mu F_\pi^2}\, 2\left[\delta m_N - \bar{\delta} m_N\right](-2)(\boldsymbol{\sigma}_i \times \boldsymbol{\sigma}_j)\cdot\hat{\boldsymbol{r}}\left(\frac{d}{dr}\frac{e^{-\mu r}}{r}\right)\psi(\mathbf{0}), \tag{37}$$

it is an easy matter to show that the matrix element we need can be reduced to a three dimensional numerical integral.

Phase Space Factors and cross section

Now finally, we come to the phase space factors and cross section. First let us define

$$\mathcal{M} \equiv \langle F|V|I\rangle. \tag{38}$$

Then we have

$$d\sigma = (2\pi)^4\langle|\mathcal{M}|_N^2\rangle\delta^{(4)}(p_1+p_2-p_\pi-p_\alpha)\frac{d^3p_\alpha d^3p_\pi 2E_\alpha}{(2\pi)^6\, 2E_\alpha 2E_\pi}\frac{4E_d^2}{4\sqrt{(p_1\cdot p_2)^2 - M_d^4}}, \tag{39}$$

$$\frac{d\sigma}{d\Omega_\pi} = \frac{\langle|\mathcal{M}|_N^2\rangle}{(2\pi)^2}\frac{E_\alpha}{E_\alpha+E_\pi}\frac{p_\pi}{2}\frac{E_d^2}{\sqrt{T_L^2M_d^2+2T_LM_d^3}} = \frac{\langle|\mathcal{M}|_N^2\rangle}{(2\pi)^2}\frac{E_\alpha p_\pi}{8k} \tag{40}$$

where the brackets indicate the average over initial spins, $p_{1,2}$ are the momenta of the two deuterons in the initial state and T_L is the deuteron kinetic energy in the laboratory frame, and p_π, E_α, E_π are cm momentum and energies. This is the angular distribution in the cm frame. For threshold production, the angular distribution is constant and so we find

$$\sigma = \frac{1}{\pi}\langle|\mathcal{M}|_N^2\rangle \frac{E_\alpha p_\pi}{8k}. \tag{41}$$

For $T_L = 228.5$ and using the operator $T^{(0)}$ we find

$$\sigma = 11.9pb \tag{42}$$

References

[1] U. van Kolck, J. A. Niskanen and G. A. Miller, Phys. Lett. B **493**, 65 (2000) [nucl-th/0006042].